MATH4060 Tutorial 4

16 February 2023

Problem 1 (Chap 6, Ex. 11)**.** *Let $f(z) = e^{az}e^{-e^z}$, where $a > 0$. Observe that in the strip $\{x + iy : |y| < \pi/2\}$ the function $f(x + iy)$ is exponentially decreasing as $|x|$ tends to infinity. Prove that*

$$\hat{f}(\xi) = \Gamma(a - 2\pi i\xi), \quad \text{for all } \xi \in \mathbb{R}.$$

For each $|y| < \pi/2$, $|f(x + iy)| = e^{ax - e^x \cos y}$ has exponential decay as $|x| \to \infty$, since $\cos y > 0$ and $a > 0$. Using a substitution $t = e^x$, we have

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\, dx = \int_{-\infty}^{\infty} e^{ax - e^x - 2\pi i\xi x}\, dx = \int_0^{\infty} t^{a-2\pi i\xi-1}e^{-t}\, dt = \Gamma(a - 2\pi i\xi).$$

Problem 2 (Chap 6, Ex. 13)**.**

(a) Prove that

$$\frac{d^2 \log \Gamma(s)}{ds^2} = \sum_{n=0}^{\infty} \frac{1}{(s+n)^2}$$

whenever s is a positive number. Show that if the left-hand side is interpreted as $(\Gamma'/\Gamma)'$, then the above formula also holds for all complex numbers $s \neq 0, -1, -2, \ldots$.

(b) Using (a), show that

$$\Gamma(s)\Gamma(s + 1/2) = \sqrt{\pi}2^{1-2s}\Gamma(2s).$$

For (a), we use Hadamard factorization

$$\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}, \qquad \Gamma(s) = e^{-\gamma s} s^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{s/n}$$

on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$ to compute that

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\gamma - \frac{1}{s} - \sum_{n=1}^{\infty} \left(\frac{1}{n+s} - \frac{1}{n}\right),$$
$$\left(\frac{\Gamma'(s)}{\Gamma(s)}\right)' = \frac{1}{s^2} + \sum_{n=1}^{\infty} \frac{1}{(n+s)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+s)^2}.$$

To justify each step above, we make use of the following facts (proofs as in Lecture 6 or Proposition 3.2 of Chapter 5): Let $f_n : \Omega \to \mathbb{C}$ be a sequence of nowhere-vanishing holomorphic functions. Suppose that $\prod f_n$ converges uniformly on compact subsets to a nowhere-vanishing (holomorphic) function f. Then

(i) $\prod(1/f_n)$ converges to $1/f$ uniformly on compact subsets,

(ii) $f'/f = \sum_{n=1}^{\infty} f_n'/f_n$, where the series converges uniformly on compact subsets.

For (b), use (a) to show that the nowhere-vanishing entire function $g(s) = \Gamma(s)\Gamma(s + 1/2)/\Gamma(2s)$ satisfies $(g'/g)' = 0$, so that $g(s) = e^{as+b}$. Substitute $s = 1$ and $s = 1/2$ to find g explicitly.

Problem 3 (Chap 6, Ex. 12)**.** *Show that*

(a) $1/|\Gamma(s)|$ *is not* $O(e^{c|s|})$ *for any* $c > 0$.

(b) there is no entire function $F(s)$ *with* $F(s) = O(e^{c|s|})$ *that has simple zeros at* $s = 0, -1, -2, \ldots$ *and that vanishes nowhere else.*

(a) Using $s\Gamma(s) = \Gamma(s+1)$, for $k \in \mathbb{N}$,

$$\Gamma(-k-1/2) = \frac{\Gamma(-(k-1)-1/2)}{(-k-1/2)} = \cdots = \frac{\sqrt{\pi}}{(-1/2)(-3/2)\cdots(-k-1/2)},$$

so that

$$\left|\frac{1}{\Gamma(-k-1/2)}\right| = \frac{(3/2)(5/2)\cdots(k+1/2)}{2\sqrt{\pi}} \geq \frac{k!}{2\sqrt{\pi}}.$$

If $1/|\Gamma(s)| = O(e^{c|s|})$ for some $c > 0$, then there exists $C > 0$ with

$$k! \leq Ce^{c(k+1/2)}$$

for all $k \in \mathbb{N}$. This is impossible since $\lim_{k\to\infty} k!/e^{c(k+1/2)} = \infty$.

(b) Suppose for a contradiction, such F exists. Then by Hadamard theorem,

$$F(s) = e^{as+b} s \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n},$$

so we can write $1/\Gamma(s) = F(s)e^{(\gamma-a)s-b}$. But the right-hand-side is $O(e^{c'|s|})$ for some $c' > 0$, which is a contradiction to (a).

Problem 4 (Chap 5, Prob. 1, Blaschke condition)**.** *Prove that if* f *is holomorphic in the unit disc, bounded and not identically zero, and* $z_1, z_2, \ldots$ *are its zeros, then*

$$\sum_n (1 - |z_n|) < \infty.$$

Fact: for $0 < a_n \leq 1$, $\prod_n a_n > 0$ if and only if $\sum_n (1 - a_n) < \infty$. To prove this, first note that the condition on a_n implies that the sequence of partial products is monotone decreasing (and bounded below by zero), so converges. If $\prod_n a_n > 0$, then $\sum_n \log a_n < \infty$ and $a_n \to 1$. Since $\lim(\log a_n)/(1-a_n) = 1$, limit comparison test shows that $\sum_n (1-a_n)$ converges. The converse is Proposition 3.1 (Chap 5) together with the condition that $a_n > 0$.

Without loss of generality, we may assume $f(0) \neq 0$ (otherwise just factor out z^m) and the number of zeros is infinite. The above observation reduces to showing that $\prod_n |z_n| > 0$. Fix $k \in \mathbb{N}$ and consider $0 < r < 1$ such that $\mathfrak{n}(r) > k$ and there is no zeros on C_r. Recall Jensen's formula:

$$\log|f(0)| - \sum_{n=1}^{\mathfrak{n}(r)} \log(|z_n|/r) = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta}|\, d\theta.$$

Boundedness of f implies that there exists a constant $M > 0$ with

$$|f(0)| \prod_{n=1}^{k} \frac{r}{|z_n|} \leq |f(0)| \prod_{n=1}^{\mathfrak{n}(r)} \frac{r}{|z_n|} \leq M.$$

Take $r \to 1^-$ to see that $\prod_{n=1}^{k} |z_n| \geq |f(0)|/M > 0$ for each $k \in \mathbb{N}$, then take $k \to \infty$.